A SPECTRAL MIMETIC LEAST-SQUARES METHOD FOR THE STOKES EQUATIONS WITH NO-SLIP BOUNDARY CONDITION

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Dedicated to Max Gunzburger's 70th birthday.

Abstract Formulation of locally conservative least-squares finite element methods (LSFEM) for the Stokes equations with the no-slip boundary condition has been a long standing problem. Existing LSFEMs that yield exactly divergence free velocities require non-standard boundary conditions [3], while methods that admit the no-slip condition satisfy the incompressibility equation only approximately [4, Chapter 7]. Here we address this problem by proving a new non-standard stability bound for the velocity-vorticity-pressure Stokes system augmented with a no-slip boundary condition. This bound gives rise to a norm-equivalent least-squares functional in which the velocity can be approximated by div-conforming finite element spaces, thereby enabling a locally-conservative approximations of this variable. We also provide a practical realization of the new LSFEM using high-order spectral mimetic finite element spaces [15] and report several numerical tests, which confirm its mimetic properties.

1. Introduction

In this paper we consider least-squares finite element methods (LSFEMs) for the velocity-vorticity-pressure (VVP) formulation of the Stokes problem

$$\begin{cases} \nabla\times\boldsymbol{\omega}+\nabla p=\boldsymbol{f} & \text{in } \Omega \\ \nabla\times\boldsymbol{u}-\boldsymbol{\omega}=0 & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \Omega \end{cases}, \tag{1}$$

where $\boldsymbol{u}$ denotes the velocity, $\boldsymbol{\omega}$ the vorticity, p the pressure and $\boldsymbol{f}$ the force per unit mass. Our main focus is on the formulation of conforming LSFEMs that are (i) locally conservative, and (ii) provably stable when the system (1) is augmented with the no-slip (velocity) boundary condition

$$\boldsymbol{u}=0 \quad \text{on } \partial\Omega. \tag{2}$$

Note that (2) is equivalent to a pair of boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{and} \quad \boldsymbol{u}\times\boldsymbol{n}=0 \quad \text{on } \partial\Omega\,, \tag{3}$$

for the normal and tangential components of the velocity field, respectively.

Formulation of conforming LSFEMs that satisfy both (i) and (ii) had been a long-standing challenge. Existing conforming methods generally fall into one of the following two categories. The LSFEMs in the first category, see e.g., [2], [8], are stable and accurate for (1) with the boundary condition (2) but satisfy $\nabla\cdot\boldsymbol{u}=0$ only approximately.

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Conversely, the LSFEMs in the second category; see, e.g., [3], [4, Chapter 7] yield exactly divergence free velocity fields but require the non-standard normal velocity, tangential vorticity boundary condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{\omega} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \, , \tag{4}$$

i.e., they specify only the first of the two velocity conditions in (3).

Thus far, achieving both stability and mass conservation with the velocity boundary condition has been only possible by switching to a non-conforming formulations such as the discontinuous LSFEMs in [6] and [7]. In this paper we address this problem by developing a new, non-standard a priory stability bound for the VVP Stokes system with (2). We refer to this bound as "non-standard" because (i) it uses an operator norm to measure the residual of the momentum equation in (1), instead of a conventional Sobolev space norm, and (ii) it employes a weak curl and grad operator in the second equation of (1). This stability bound gives rise to a norm-equivalent functional, which can be discretized by using div-conforming elements for the velocity fild. In so doing we are able to obtain a LSFEM that is both locally conservative and stable for (1)–(2).

We have organized the rest of the paper as follows. Section 2 introduces notation and some necessary background results. In Section 3 a non-standard stability bound will be given. In Section 4 the associated variational formulation will be presented. Section 5 introduces conforming finite dimensional subspaces which respect the properties of the exact sequences. In this section all operations on polynomials will be represented by operations on their expansion coefficients. Results of the mimetic least-squares spectral element will be presented. Concluding remarks and future work are discussed in Section 7.

2. Preliminaries

In what follows $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded open region with Lipschitz boundary $\partial\Omega$. We recall the space $L^2(\Omega)$ of all square integrable functions with norm and inner product denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$, respectively, and its subspace $L^2_0(\Omega)$ of all square integrable functions with a vanishing mean. The spaces $H(grad,\Omega)$, $H(curl,\Omega)$ and $H(div,\Omega)$ contain square integrable functions whose gradient, curl and divergence are also square integrable. When equipped with the graph norms

$$\|q\|_{grad} := \|q\|_0^2 + \|\nabla q\|_0^2, \;\; \|\boldsymbol{\xi}\|^2_{curl} := \|\boldsymbol{\xi}\|_0^2 + \|\nabla \times \boldsymbol{\xi}\|_0^2 \;, \text{ and } \|\boldsymbol{v}\|^2_{div} := \|\boldsymbol{v}\|_0^2 + \|\nabla \cdot \boldsymbol{v}\|_0^2 \;,$$

the spaces $H(grad,\Omega)$, $H(curl,\Omega)$ and $H(div,\Omega)$ are Hilbert spaces.

We will also need the factor space $H(grad,\Omega)/\mathbb{R}$, which contains equivalence classes of functions in $H(grad,\Omega)$ differing by a constant, and the subspaces

$$H_0(curl,\Omega) = \{\boldsymbol{v} \in H(curl,\Omega) \,|\, \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \,,$$

$$H_0(div,\Omega) = \{\boldsymbol{v} \in H(div,\Omega) \,|\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \;,$$

of $H(curl,\Omega)$ and $H(div,\Omega)$, respectively containing functions whose tangential and normal traces vanish on the boundary. The Poincare inequalities

$$\|q\|_0 \le C\|\nabla q\|_0, \quad \|\boldsymbol{\xi}\|_0 \le C\,\{\|\nabla \times \boldsymbol{\xi}\|_0 + \|\nabla \cdot \boldsymbol{\xi}\|_0\}\,, \quad \|\boldsymbol{v}\|_0 \le C\,\{\|\nabla \cdot \boldsymbol{v}\|_0 + \|\nabla \times \boldsymbol{v}\|_0\}\,, \tag{5}$$

which hold for all $q \in H(grad,\Omega)/\mathbb{R}$, $\boldsymbol{\xi} \in H_0(curl,\Omega)$ and $\boldsymbol{v} \in H_0(div,\Omega)$, respectively, imply that the associated semi-norms are norms on these spaces.

The spaces $H(grad,\Omega)$, $H(curl,\Omega)$ and $H(div,\Omega)$, and the associated spaces $H(grad,\Omega)/\mathbb{R}$, $H_0(curl,\Omega)$, and $H_0(div,\Omega)$ provide the domains for the gradient, divergence and curl operators. We denote the ranges of these operators by $\mathcal{R}(\star)$, resp., $\mathcal{R}_0(\star)$ where $\star \in \{\nabla, \nabla\times, \nabla\cdot\}$. For instance, $\mathcal{R}(\nabla\times)$ is the range of curl acting on $H(curl,\Omega)$ and $\mathcal{R}_0(\nabla\cdot)$ is the range of divergence acting on $H_0(div,\Omega)$. Likewise, $\mathcal{N}(\star)$, resp. $\mathcal{N}_0(\star)$ are the nullspaces of these operators in the appropriate function spaces.

The vector identities $\nabla\times(\nabla) = 0$ and $\nabla\cdot(\nabla\times) \equiv 0$ imply that $\mathcal{R}_0(\nabla) \subseteq \mathcal{N}_0(\nabla\times)$, $\mathcal{R}_0(\nabla\times) \subseteq \mathcal{N}_0(\nabla\cdot)$, and $\mathcal{R}(\nabla) \subseteq \mathcal{N}(\nabla\times)$ and $\mathcal{R}(\nabla\times) \subseteq \mathcal{N}(\nabla\cdot)$. In this paper we assume that Ω is such that

$$\mathcal{R}_0(\nabla) = \mathcal{N}_0(\nabla\times),\ \mathcal{R}(\nabla) = \mathcal{N}(\nabla\times) \quad \text{and} \quad \mathcal{R}_0(\nabla\times) = \mathcal{N}_0(\nabla\cdot),\ \mathcal{R}(\nabla\times) = \mathcal{N}(\nabla\cdot). \tag{6}$$

A sufficient condition[1] for (6) to hold is for Ω to be a contractible, or star-shaped. This result is known as general Poincare lemma [19, p.69]

2.1. **Adjoint operators and decompositions.** The proof of the non-standard stability bound in Section 3 uses on orthogonal decompositions of $H_0(div,\Omega)$ and $H(div,\Omega)$. Since $\mathcal{R}_0(\nabla\times)$ is a closed subspace of $H_0(div,\Omega)$ and $\mathcal{R}(\nabla\times)$ is a closed subspace of $H(div,\Omega)$, assumption (6) implies that

$$H_0(div,\Omega) = \mathcal{R}_0(\nabla\times)\oplus\mathcal{R}_0(\nabla\times)^\perp = \mathcal{N}_0(\nabla\cdot)\oplus\mathcal{N}_0(\nabla\cdot)^\perp$$
$$H(div,\Omega) = \mathcal{R}(\nabla\times)\oplus\mathcal{R}(\nabla\times)^\perp = \mathcal{N}(\nabla\cdot)\oplus\mathcal{N}(\nabla\cdot)^\perp\,.$$

For instance, the first decomposition means that every $\boldsymbol{u}\in H_0(div,\Omega)$ can be written as

$$\boldsymbol{u} = \boldsymbol{u}_N + \boldsymbol{u}_{N^\perp}\ . \tag{7}$$

where $\boldsymbol{u}_N \in \mathcal{N}_0(\nabla\cdot)$ and $\boldsymbol{u}_{N^\perp}\in\mathcal{N}(\nabla\cdot)^\perp$. The nullspace component $\boldsymbol{u}_N = \nabla\times\boldsymbol{\xi}$ solves the variational equation: seek $\boldsymbol{\xi}\in H_0(curl,\Omega)$ and $\mu\in H_0(grad,\Omega)$ such that

$$\begin{array}{rcll} (\nabla\times\boldsymbol{\xi},\nabla\times\boldsymbol{\zeta})_0 + (\boldsymbol{\zeta},\nabla\mu)_0 &=& (\boldsymbol{u},\nabla\times\boldsymbol{\zeta})_0 & \forall\boldsymbol{\zeta}\in H_0(curl,\Omega),\\ (\boldsymbol{\xi},\nabla\lambda)_0 &=& 0 & \forall\lambda\in H_0(grad,\Omega) \end{array} \tag{8}$$

The orthogonal complement component $\boldsymbol{u}_{N^\perp}$ solves a similar mixed problem: seek $\boldsymbol{u}_{N^\perp}\in H_0(div,\Omega)$ and $\phi\in L^2_0(\Omega)$ such that

$$\begin{array}{rcll} (\boldsymbol{u}_{N^\perp},\boldsymbol{v})_0 + (\phi,\nabla\cdot\boldsymbol{v})_0 &=& 0 & \forall\boldsymbol{v}\in H_0(div,\Omega),\\ (\nabla\cdot\boldsymbol{u}_{N^\perp},\varphi)_0 &=& (\nabla\cdot\boldsymbol{u},\varphi) & \forall\varphi\in L^2_0(\Omega) \end{array} \tag{9}$$

For a given $\phi\in L^2_0(\Omega)$ the first equation in (9) induces a mapping $\phi\mapsto\boldsymbol{u}_{N^\perp}$ which we call a "weak" gradient ∇^* of ϕ. Succinctly, $\nabla^* : L^2_0(\Omega)\mapsto H_0(div,\Omega)$ according to

$$(\nabla^*\phi,\boldsymbol{v})_0 := (\phi,-\nabla\cdot\boldsymbol{v})_0\ ,\quad \forall\boldsymbol{v}\in H_0(div,\Omega)\,. \tag{10}$$

One can show that under assumption (6) there holds $\mathcal{N}_0(\nabla\cdot)^\perp = \mathcal{R}_0(\nabla^*)$ and so,

$$H_0(div,\Omega) = \mathcal{R}_0(\nabla\times)\oplus\mathcal{R}_0(\nabla^*)\,. \tag{11}$$

In other words, the decomposition assumes the form

$$\boldsymbol{u} = \nabla\times\boldsymbol{\xi} + \nabla^*\phi \tag{12}$$

where $\boldsymbol{\xi}\in H_0(curl,\Omega)$ and $\phi\in L^2_0(\Omega)$. In particular, $\phi\in L^2_0(\Omega)$ satisfies $\nabla^*\phi\cdot\boldsymbol{n} = 0$.

We will also need a "weak" version of the curl operator $\nabla^*\times\boldsymbol{u} : H(div,\Omega)\mapsto H(curl,\Omega)$ defined by

$$(\nabla^*\times\boldsymbol{u},\boldsymbol{\xi})_0 := (\boldsymbol{u},\nabla\times\boldsymbol{\xi})_0\ ,\quad \forall\boldsymbol{\xi}\in H(curl,\Omega)\,. \tag{13}$$

The operator $\nabla^*\times$ enforces weakly the boundary condition $\boldsymbol{u}\times\boldsymbol{n} = 0$ on its argument.

And we need a "weak" version of the gradient operator $\nabla^* p : L^2(\Omega)\mapsto H(div,\Omega)$ defined by

$$(\nabla^* p,\boldsymbol{u})_0 := (p,-\nabla\cdot\boldsymbol{u})_0\ ,\quad \forall\boldsymbol{u}\in H(div,\Omega)\,. \tag{14}$$

[1]The first identity holds if Ω has no loops, whereas the second identity holds if Ω has no holes.

2.2. **Weak norms and seminorms.** The a priory stability bound for (1) that provides the foundation for our mimetic LSFEM requires nonstandard norms and seminorms for functions in $H(div,\Omega)$, $H(curl,\Omega)$ and $L^2(\Omega)$. First, for any $\boldsymbol{u} \in H(div,\Omega)$ we define

$$\|\boldsymbol{u}\|_D = \sup_{\boldsymbol{v}\in H(div,\Omega)} \frac{(\boldsymbol{u},\boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} \,. \tag{15}$$

Second, for any $\boldsymbol{\xi} \in H(curl,\Omega)$ we define

$$\|\nabla\times\boldsymbol{\omega}\|_{N_0} := \sup_{\boldsymbol{v}\in\mathcal{N}_0(\nabla\cdot)} \frac{(\nabla\times\boldsymbol{\omega},\boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} \,. \tag{16}$$

Lastly, for any $p \in L^2(\Omega)$ we define

$$\|\nabla^* p\|_{N_0^\perp} := \sup_{\boldsymbol{v}\in\mathcal{N}_0(\nabla\cdot)^\perp} \frac{(p,\nabla\cdot v)_0}{\|\boldsymbol{v}\|_{div}} \,. \tag{17}$$

Norm (15) is simply the norm on the dual space $H(div,\Omega)'$. Insofar as (16) is concerned taking sup over the nullspace $\mathcal{N}_0(\nabla\cdot)$ implies that

$$\|\nabla\times\boldsymbol{\omega}\|_{N_0} \le \|\nabla\times\boldsymbol{\omega}\|_0$$

and so we call (16) weak curl semi norm. Finally, it is easy to see that (17) satisfies

$$\|\nabla^* p\|_{N_0^\perp} \le \|\nabla^* p\|_0$$

and we will refer to it as the weak norm on $L^2(\Omega)$.

3. Non-standard stability bound

In this section we establishes a priori stability bound for the VVP Stokes system (1) with the no-slip boundary condition (3). The proof draws upon the techniques in [3] with one important distinction. Stability proof in that paper relies on the orthogonality between $\nabla\times\boldsymbol{\omega}$ and $\nabla^* p$ when $\boldsymbol{\omega}$ has a vanishing tangential component, i.e., the fact that

$$(\nabla\times\omega,\nabla^* p) = 0 \quad \forall\boldsymbol{\omega}\in H_0(curl,\Omega) \quad \text{and} \quad p\in L^2(\Omega)\,. \tag{18}$$

This implies a trivial lower bound (in fact an identity) for the L^2-norm of the residual of the momentum equation

$$\|\nabla\times\boldsymbol{\omega}+\nabla^* p\|_0 \ge \|\nabla\times\boldsymbol{\omega}\|_0 + \|\nabla^* p\|_0 \,, \tag{19}$$

which represents a key juncture in the proof.

However, for the case of the no-slip boundary condition of interest to us, $\boldsymbol{\omega}\in H(curl,\Omega)$ and $\nabla\times\boldsymbol{\omega}$ is not orthogonal to $\nabla^* p$. As a result, (18), resp. (19) do not hold. Nonetheless, the following theorem demonstrates that a lower bound similar to (19) can be established in terms of appropriate weak norms and semi norms.

Theorem 1. *For all $\omega\in H(curl,\Omega)$ and all $p\in L^2(\Omega)$ there holds*

$$\|\nabla\times\boldsymbol{\omega}+\nabla^* p\|_D \ge \frac{1}{2}\left(\|\nabla\times\boldsymbol{\omega}\|_{N_0} + \|\nabla^* p\|_{N^\perp}\right) \,. \tag{20}$$

Proof. In order to bound the dual norm (15) from below by the weak curl-seminorm we restrict the supremum to functions in $\mathcal{N}_0(\nabla\cdot)$ and use the fact that $\nabla^* p\in\mathcal{N}_0^\perp(\nabla\cdot)$, see (14). As a result,

$$\begin{aligned}\|\nabla\times\boldsymbol{\omega}+\nabla^* p\|_D &= \sup_{\boldsymbol{v}\in H(div,\Omega)} \frac{(\nabla\times\boldsymbol{\omega}+\nabla^* p,\boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} \ge \sup_{\boldsymbol{v}\in\mathcal{N}_0(\nabla\cdot)} \frac{(\nabla\times\boldsymbol{\omega}+\nabla^* p,\boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} \\ &\overset{\nabla^* p\in\mathcal{N}_0^\perp(\nabla\cdot)}{=} \sup_{\boldsymbol{v}\in\mathcal{N}_0(\nabla\cdot)} \frac{(\nabla\times\boldsymbol{\omega},\boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} = \|\nabla\times\boldsymbol{\omega}\|_{N_0} \,.\end{aligned} \tag{21}$$

Similarly, to bound the dual norm in terms of the weak L^2 norm we restrict the supremum to the orthogonal complement $\mathcal{N}_0(\nabla\cdot)^\perp$. From (12) we know that all elements of $\mathcal{N}_0(\nabla\cdot)^\perp$ are given by $\nabla^*\phi$ for $\phi \in L^2_0(\Omega)$. From (13) we have that

$$0 = (\nabla^* \times \nabla^* \phi, \boldsymbol{\omega})_0 = (\nabla^*\phi, \nabla \times \boldsymbol{\omega})_0 \ ,$$

where again $\nabla^*\times$ weakly enforces $\nabla^*\phi \times \boldsymbol{n} = 0$:

$$\begin{aligned} \|\nabla \times \boldsymbol{\omega} + \nabla^* p\|_D &= \sup_{\boldsymbol{v}\in H(div,\Omega)} \frac{(\nabla \times \boldsymbol{\omega} + \nabla^* p, \boldsymbol{v})_0}{\|\boldsymbol{v}\|_{div}} \geq \sup_{\boldsymbol{v}\in \mathcal{N}_0(\nabla\cdot)^\perp} \frac{(\nabla \times \boldsymbol{\omega} + \nabla^* p, \boldsymbol{v})_0}{\|v_p\|_{div}} \\ &\overset{\nabla\times\boldsymbol{\omega}\perp\mathcal{N}_0(\nabla\cdot)^\perp}{=} \sup_{\boldsymbol{v}\in \mathcal{N}_0(\nabla\cdot)^\perp} \frac{(\nabla^* p, \boldsymbol{v})_0}{\|v\|_{div}} = \|\nabla^* p\|_{N_0^\perp} \ . \end{aligned} \tag{22}$$

Combination of these two bounds proves the theorem:

$$\begin{aligned} \|\nabla \times \boldsymbol{\omega} + \nabla^* p\|_D &= \frac{1}{2}\|\nabla \times \boldsymbol{\omega} + \nabla^* p\|_D + \frac{1}{2}\|\nabla \times \boldsymbol{\omega} + \nabla^* p\|_D \\ &\overset{(21,22)}{\geq} \frac{1}{2}\|\nabla \times \boldsymbol{\omega}\|_{N_0} + \frac{1}{2}\|\nabla^* p\|_{N^\perp} \ . \end{aligned} \tag{23}$$

□

Remark 1. *Unlike the case of the nonstandard normal velocity-tangential vorticity boundary condition considered in [3], in which* (19) *holds with an identity, Theorem 1 can only bound a dual norm of the momentum equation from below by weaker semi norms of the vorticity and the pressure. As mentioned at the beginning of this section, the reason for this is the lack of orthogonality between $\nabla \times \boldsymbol{\omega}$ and $\nabla^* p$.*

To state the main result of this section it is convenient to introduce the "weak" curl norm

$$\|\boldsymbol{\omega}\|^2_{N_0} = \|\boldsymbol{\omega}\|^2_0 + \|\nabla \times \boldsymbol{\omega}\|^2_{N_0}.$$

Theorem 2. *There exists a constant $C > 0$ such that*

$$\|\nabla \times \omega + \nabla^* p\|^2_D + \|\omega - \nabla^* \times \boldsymbol{u}\|^2_0 + \|\nabla \cdot \boldsymbol{u}\|^2_0 \geq C\left\{\|\omega\|^2_{N_0} + \|\boldsymbol{u}\|_{div} + \|\nabla^* p\|^2_{N_0^\perp}\right\} \ .$$

for every $\omega \in H(curl,\Omega)$, $\boldsymbol{u} \in H_0(div,\Omega)$ and $p \in L^2(\Omega)$.

Proof. The proof follows the ideas in [3]. We have

$$\|\nabla^* \times \boldsymbol{u} - \omega\|^2_0 = \|\nabla^* \times \boldsymbol{u}\|^2_0 + \|\omega\|^2_0 - 2(\nabla^* \times \boldsymbol{u}, \omega) \tag{24}$$

where $\nabla^*\times$ is the weak curl[2] defined in (13). To bound the last term we split it in two equal parts. On the one hand, the Cauchy-Schwartz inequality gives the bound

$$(\nabla^* \times \boldsymbol{u}, \omega) \leq \|\nabla^* \times \boldsymbol{u}\|_0 \|\boldsymbol{\omega}\|_0 \,.$$

On the other hand, using (12) and (13)

$$(\nabla^* \times \boldsymbol{u}, \boldsymbol{\omega}) = (\nabla^* \times (\boldsymbol{u}_N + \boldsymbol{u}_{N^\perp}), \boldsymbol{\omega}) = (\nabla^* \times \boldsymbol{u}_N, \boldsymbol{\omega}) = (\boldsymbol{u}_N, \nabla \times \boldsymbol{\omega})\,,$$

and since $\boldsymbol{u}_N \in N_0(\nabla\cdot)$ definition (16) of the weak curl semi norm implies

$$(\boldsymbol{u}_N, \nabla \times \boldsymbol{\omega}) \leq \|\boldsymbol{u}_N\|_0 \|\nabla \times \boldsymbol{\omega}\|_{N_0} \leq \|\boldsymbol{u}\|_0 \|\nabla \times \boldsymbol{\omega}\|_{N_0} \,.$$

Using these inequalities in (24) gives the bound

$$\|\nabla^* \times \boldsymbol{u} - \omega\|^2_0 \geq \|\nabla^* \times \boldsymbol{u}\|^2_0 + \|\omega\|^2_0 - \|\nabla^* \times \boldsymbol{u}\|_0 \|\omega\|_0 - \|\boldsymbol{u}\|_0 \|\nabla \times \omega\|_{N_0} \ .$$

[2] We recall that this operator enforces $\boldsymbol{u} \times \boldsymbol{n} = 0$ in a weak, variational sense. As a result, the first part of the no-slip condition (3) is enforced strongly through $\boldsymbol{u} \in H_0(div,\Omega)$, while the second part is enforced weakly through the definition of $\nabla^*\times$.

Using the ϵ-inequality for the last two terms gives

$$\|\nabla^* \times \boldsymbol{u} - \omega\|_0^2 \geq \left(1 - \frac{\delta}{2}\right)\|\nabla^* \times \boldsymbol{u}\|_0^2 + \left(1 - \frac{1}{2\delta}\right)\|\omega\|_0^2 - \frac{\epsilon}{2}\|\boldsymbol{u}\|_0^2 - \frac{1}{2\epsilon}\|\nabla \times \omega\|_{N_0}^2 \; . \tag{25}$$

Using Poincaré-Friedrichs inequality [4, Theorems A.10-A.11]

$$\|\nabla^* \times \boldsymbol{u}\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2 \geq \frac{1}{C_P^2}\|\boldsymbol{u}\|_0^2 \, , \tag{26}$$

gives

$$\|\nabla^* \times \boldsymbol{u} - \omega\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2 \geq \frac{1}{2}\left(1-\delta\right)\|\nabla^* \times \boldsymbol{u}\|_0^2 + \frac{1}{2}\|\nabla \cdot \boldsymbol{u}\|_0^2 + \left(1 - \frac{1}{2\delta}\right)\|\omega\|_0^2 + \frac{1}{2}\left(\frac{1}{C_P^2} - \epsilon\right)\|\boldsymbol{u}\|_0^2 - \frac{1}{2\epsilon}\|\nabla \times \omega\|_{N_0}^2 \; . \tag{27}$$

Adding β times the momentum equation yields and using Theorem 1

$$\beta\|\nabla \times \omega + \nabla^* p\|_D^2 + \|\nabla^* \times \boldsymbol{u} - \omega\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2 \geq \frac{1}{2}\left(1-\delta\right)\|\nabla^* \times \boldsymbol{u}\|_0^2 + \frac{1}{2}\|\nabla \cdot \boldsymbol{u}\|_0^2 + \left(1 - \frac{1}{2\delta}\right)\|\omega\|_0^2 + \frac{1}{2}\left(\frac{1}{C_P^2} - \epsilon\right)\|\boldsymbol{u}\|_0^2 + \frac{1}{2}\left(\beta - \epsilon\right)\|\nabla \times \omega\|_{N_0}^2 + \frac{\beta}{2}\|\nabla^* p\|_{N_0^\perp}^2 \; . \tag{28}$$

For the specific choice $\epsilon = 1/C_P^2$, $\delta = 2/3$ and $\beta = 1 + 1/C_P^2$

$$\begin{aligned}(1 + \frac{1}{C_P^2})\|\nabla \times \omega + \nabla^* p\|_D^2 + \|\nabla^* \times \boldsymbol{u} - \omega\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2 \\ \geq \frac{1}{6}\|\nabla^* \times \boldsymbol{u}\|_0^2 + \frac{1}{2}\|\nabla \cdot \boldsymbol{u}\|_0^2 + \frac{1}{4}\|\omega\|_0^2 + \frac{1}{2}\|\nabla \times \omega\|_{N_0}^2 + \frac{1}{2C_P^2}(C_P^2 + 1)\|\nabla^* p\|_{N_0^\perp}^2 \\ \geq \min\left\{\frac{1}{6}, \frac{1}{2C_P^2}(C_P^2 + 1)\right\}\left(\|\omega\|_{curl}^2 + \|\boldsymbol{u}\|_{H_0(div,\Omega)}^2 + \|\nabla^* p\|_{N_0^\perp}^2\right) \; .\end{aligned} \tag{29}$$

The theorem follows from

$$\|\omega - \nabla^* \times \boldsymbol{u}\|_0^2 + \|\nabla \times \omega + \nabla^* p\|_D^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2 \geq \frac{C_P^2}{1 + C_P^2}\left[(1 + \frac{1}{C_P^2})\|\nabla \times \omega + \nabla^* p\|_D^2 + \|\nabla^* \times \boldsymbol{u} - \omega\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2\right] \; . \tag{30}$$

□

4. Variational formulation

Based on the coercivity result in Theorem 2 we define the following least-squares function for $(\boldsymbol{\xi}, \boldsymbol{v}, q) \in \mathcal{X} = H(curl, \Omega) \times H_0(div, \Omega) \times L^2(\Omega)$ and $\boldsymbol{f} \in H(div, \Omega)$

$$\mathcal{J}(\boldsymbol{\xi}, \boldsymbol{v}, q; \boldsymbol{f}) := \frac{1}{2}\left\{\|\nabla^* \times \boldsymbol{v} - \boldsymbol{\xi}\|_0^2 + \|\nabla \times \boldsymbol{\xi} + \nabla^* q - \boldsymbol{f}\|_D^2 + \|\nabla \cdot \boldsymbol{v}\|_0^2\right\} \; . \tag{31}$$

The unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of the Stokes problem is then given by

$$(\boldsymbol{\omega}, \boldsymbol{u}, p) = \arg \min_{(\boldsymbol{\xi}, \boldsymbol{v}, q) \in \mathcal{X}} \mathcal{J}(\boldsymbol{\xi}, \boldsymbol{v}, q; \boldsymbol{f}) \; . \tag{32}$$

From the definition of the operator norm (15) it follows that for $u \in H(div, \Omega)$, $\|u\|_D = 0$ iff $(\boldsymbol{u}, \boldsymbol{v})_0 = 0$ for all $\boldsymbol{v} \in H(div, \Omega)$. Every $\boldsymbol{v} \in H(div, \Omega)$ has a decomposition $\boldsymbol{v} = \nabla \times \boldsymbol{\xi} + \nabla^* q$ with $\boldsymbol{\xi} \in H(curl, \Omega)$ and $q \in L^2(\Omega)$, although this decomposition is *not* orthogonal

$$\|\nabla \times \boldsymbol{\omega} + \nabla^* p - \boldsymbol{f}\|_D = 0 \quad \Longleftrightarrow \quad (\nabla \times \boldsymbol{\omega} + \nabla^* p - \boldsymbol{f}, \nabla \times \boldsymbol{\xi} + \nabla^* q)_0 = 0 \,,$$

for all $\boldsymbol{\xi} \in H(curl, \Omega)$ and $q \in L^2(\Omega)$. Taking variations of $\|\boldsymbol{\omega} - \nabla^* \times \boldsymbol{u}\|_0^2$ gives

$$(\boldsymbol{\omega} - \nabla^* \times \boldsymbol{u}, \boldsymbol{\xi} - \nabla^* \times \boldsymbol{v})_0 = 0 \,,$$

for all $\boldsymbol{\xi} \in H(curl, \Omega)$ and $\boldsymbol{v} \in H_0(div, \Omega)$. Finally, conservation of mass is satisfied if $\|\nabla \cdot \boldsymbol{u}\|_0^2 = 0$ which implies that

$$(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_0 = 0 \,,$$

for all $\boldsymbol{v} \in H_0(div, \Omega)$. If we collect all these conditions for a minimizer we obtain

$$\begin{aligned} (\nabla \times \boldsymbol{\omega} + \nabla^* p, \nabla \times \boldsymbol{\xi})_0 + (\boldsymbol{\omega} - \nabla^* \times \boldsymbol{u}, \boldsymbol{\xi})_0 &= (\boldsymbol{f}, \nabla \times \boldsymbol{\xi})_0 && \forall \boldsymbol{\xi} \in H(curl, \Omega) \\ (\nabla^* \times \boldsymbol{u} - \boldsymbol{\omega}, \nabla^* \times \boldsymbol{v})_0 + (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_0 &= 0 && \forall \boldsymbol{v} \in H_0(div, \Omega) \\ (\nabla \times \boldsymbol{\omega} + \nabla^* p, \nabla^* q)_0 &= (\boldsymbol{f}, \nabla^* q)_0 && \forall q \in L^2(\Omega) \end{aligned} \tag{33}$$

Using integration by parts, using (13) and (14), we have: Find $(\boldsymbol{\omega}, \boldsymbol{u}, p) \in \mathcal{X}$ such that

(34)
$$\begin{aligned} (\nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\xi})_0 + (\boldsymbol{\omega}, \boldsymbol{\xi})_0 - (\boldsymbol{u}, \nabla \times \boldsymbol{\xi})_0 &= (\boldsymbol{f}, \nabla \times \boldsymbol{\xi})_0 && \forall \boldsymbol{\xi} \in H(curl, \Omega) \\ -(\nabla \times \boldsymbol{\omega}, \boldsymbol{v})_0 + (\nabla^* \times \boldsymbol{u}, \nabla^* \times \boldsymbol{v})_0 + (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_0 &= 0 && \forall \boldsymbol{v} \in H_0(div, \Omega) \\ (\nabla^* p, \nabla^* q)_0 &= (-\nabla \cdot \boldsymbol{f}, q)_0 && \forall q \in L^2(\Omega) \end{aligned}$$

The well-posedness result, Theorem 2, is inherited on conforming subspaces $\mathbf{C}^h \subset H(curl, \Omega)$, $\mathbf{D}_0^h \subset H_0(div, \Omega)$ and $S^h \subset L^2(\Omega)$. The variational equation then becomes: Find $(\boldsymbol{\omega}^h, \boldsymbol{u}^h, p^h) \in \mathcal{X}^h = \mathbf{C}^h \times \mathbf{D}_0^h \times S^h$ such that

(35)
$$\begin{aligned} \left(\nabla \times \boldsymbol{\omega}^h, \nabla \times \boldsymbol{\xi}^h\right)_0 + \left(\boldsymbol{\omega}^h, \boldsymbol{\xi}^h\right)_0 - \left(\boldsymbol{u}^h, \nabla \times \boldsymbol{\xi}^h\right)_0 &= \left(\boldsymbol{f}, \nabla \times \boldsymbol{\xi}^h\right)_0 && \forall \boldsymbol{\xi}^h \in \mathbf{C}^h \\ -\left(\nabla \times \boldsymbol{\omega}^h, \boldsymbol{v}^h\right)_0 + \left(\nabla^* \times \boldsymbol{u}^h, \nabla^* \times \boldsymbol{v}^h\right)_0 + \left(\nabla \cdot \boldsymbol{u}^h, \nabla \cdot \boldsymbol{v}^h\right)_0 &= 0 && \forall \boldsymbol{v}^h \in \mathbf{D}_0^h \\ \left(\nabla^* p^h, \nabla^* q^h\right)_0 &= \left(-\nabla \cdot \boldsymbol{f}, q^h\right)_0 && \forall q^h \in S^h \end{aligned}$$

The next section will be devoted to the construction of such finite dimensional conforming subspaces.

5. Mimetic spectral element method

Mimetic methods aim to decompose partial differential operators in a purely topological part and a metric dependent part. The advantage of the purely topological description is that it is independent of the size or shape of the mesh or the order of the approximation. We call such relations *exact*. The orthogonal decomposition of $\mathbf{D}_0^h$ into divergence-free vector fields and irrotational vector fields is fully represented by the topological part. Approximation takes place in the metric dependent part. In this section we will present the spectral element basis functions which span the finite dimensional spaces $\mathbf{C}^h$, $\mathbf{D}^h$ and S^h.

Similar ideas can be found in Tonti, [20, 21], Bossavit, [10, 11], Desbrun et al., [13], Bochev and Hyman, [5], Lipnikov et al, [17], Seslija et al., [18], Bonelle and Ern, [9], Lemoine et al., [16], Kreeft et al., [15] and references therein.

5.1. **One dimensional basis spectral basis functions.** Consider the interval $[-1,1] \subset \mathbb{R}$ and the Legendre polynomials, $L_N(\xi)$ of degree N, $\xi \in [-1,1]$. The $(N+1)$ roots, ξ_i, of the polynomial $(1-\xi^2)L_N'(\xi)$ satisfy $-1 \le \xi_i \le 1$. Here $L_N'(\xi)$ is the derivative of the Legendre polynomial. The zeros are called the *Gauss-Lobatto-Legendre (GLL) points*. Let $h_i(\xi)$ be the Lagrange polynomial through the GLL points such that

$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \neq j \end{cases} \qquad i,j = 0,\dots N\ . \tag{36}$$

The explicit form of the Lagrange polynomials in terms of the Legendre polynomials is given by

$$h_i(\xi) = \frac{(1-\xi^2)L_N'(\xi)}{N(N+1)L_N(\xi_i)(\xi_i - \xi)}\ . \tag{37}$$

Let $f(\xi)$ be defined for $\xi \in [-1,1]$ by

$$f(\xi) = \sum_{i=0}^{N} a_i h_i(\xi)\ . \tag{38}$$

Using property (36) we see that $f(\xi_j) = a_j$, so the expansion coefficients in (38) coincide with the value of f in the GLL nodes. We will refer to this expansion as a *nodal expansion*. The basis functions $h_i(\xi)$ are polynomials of degree N.

From the nodal basis functions we define the functions $e_i(\xi)$ by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\,\mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi)\ . \tag{39}$$

The functions $e_i(\xi)$ are polynomials of degree $(N-1)$. These polynomials satisfy, [14, 15]

$$\int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \neq j \end{cases} \qquad i,j = 1,\dots N\ . \tag{40}$$

Let a function $f(\xi)$ be expanded in these functions

$$f(\xi) = \sum_{i=1}^{N} b_i e_i(\xi)\ , \tag{41}$$

then using (40)

$$\int_{\xi_{j-1}}^{\xi_j} f(\xi) = b_j\ .$$

So the expansion coefficients b_i coincide with the integral of f over the edge $[\xi_{i-1}, \xi_i]$. We will call these basis functions *edge functions* and refer to the expansion (41) as an *edge expansion*, see for instance [1, 15] for examples of nodal and edge expansions.

Let $f(\xi)$ be expanded in terms Lagrange polynomials as in (38), then the derivative of f is given by, [14, 15]

$$f'(\xi) = \sum_{i=0}^{N} a_i h_i'(\xi) = \sum_{i=1}^{N} (a_i - a_{i-1}) e_i(\xi)\ . \tag{42}$$

Remark 2. *Note that the set of polynomials $\{h_i'\}$, $i = 0,\dots,N$ is linear dependent and therefore does not form a basis, while the set $\{e_i\}$, $i = 1,\dots,N$ is linear independent and therefore forms a basis for the derivatives of the nodal expansion (38).*

For all integrals we use Gauss-Lobatto integration

$$\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi \approx \sum_{i=0}^{N} f(\xi_i)w_i\ , \tag{43}$$

where the Gauss-Lobatto weight are given by

$$w_i = \begin{cases} \frac{2}{N(N+1)} & \text{if } i = 0 \text{ and } i = N \\ \\ \frac{2}{N(N+1)L_N^2(\xi_i)} & \text{if } i = 1,\ldots,N-1 \end{cases} \tag{44}$$

Gauss-Lobatto integration is exact for polynomials of degree $2N-1$, see [12].

5.2. **Two dimensional expansions.** The decomposition of any vector field in $H_0(div,\Omega)$ into a divergence-free part and curl-free part, (2.1), is pivotal to the analysis in Section 2. Consider $[-1,1]^2 \subset \mathbb{R}^2$. We will use tensor products of nodal and edge expansions to construct conforming finite dimensional subspaces $\mathbf{C}_0^h$, $\mathbf{D}_0^h$ and S_0^h, and $\mathbf{C}^h$, $\mathbf{D}^h$ of $H_0(curl,\Omega)$, $H_0(div,\Omega)$, $L_0^2(\Omega)$ and $H(curl,\Omega)$ and $H(div,\Omega)$, respectively. Let (ξ_i,η_j) the GLL points in ξ- and η-direction. We will first describe the finite dimensional spaces and the primal vector operations, $\nabla\times$ and $\nabla\cdot$, between these space.

5.2.1. *Spaces and primal vector operators.* Let the space $\mathbf{C}^h$ consist of the span of $\{h_i(\xi)h_j(\eta)\}$, $i,j=0,\ldots,N$. So any function $\boldsymbol{\omega}^h(\xi,\eta) \in \mathbf{C}^h$ can be written as

$$\boldsymbol{\omega}^h(\xi,\eta) = \sum_{i=0}^{N}\sum_{j=0}^{N} \omega_{i,j} h_i(\xi)h_j(\eta) = [h_0(\xi)h_0(\eta)\ \ldots\ h_N(\xi)h_N(\eta)] \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} . \tag{45}$$

From (36) it follows that $\omega_{i,j} = \boldsymbol{\omega}(\xi_i,\eta_j)$. We obtain $\mathbf{C}_0^h$ from $\mathbf{C}^h$ by setting the degrees of freedom on the boundary to zero, so for $\psi^h \in \mathbf{C}_0^h$ we have the expansion

$$\psi^h(\xi,\eta) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \psi_{i,j} h_i(\xi)h_j(\eta) = [h_1(\xi)h_1(\eta)\ \ldots\ h_{N-1}(\xi)h_{N-1}(\eta)] \begin{bmatrix} \psi_{1,1} \\ \vdots \\ \psi_{N-1,N-1} \end{bmatrix} . \tag{46}$$

If we apply the 2D curl to $\boldsymbol{\omega}^h$ we obtain, using (42)

$$\begin{aligned} \nabla\times\boldsymbol{\omega}^h &= \begin{pmatrix} \sum_{i=0}^{N}\sum_{j=1}^{N}(\omega_{i,j}-\omega_{i,j-1})h_i(\xi)e_j(\eta) \\ \sum_{i=1}^{N}\sum_{j=0}^{N}(\omega_{i-1,j}-\omega_{i,j})e_i(\xi)h_j(\eta) \end{pmatrix} \\ &= \begin{bmatrix} h_0(\xi)e_1(\eta) & \ldots & h_N(\xi)e_N(\eta) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & e_1(\xi)h_0(\eta) & \ldots & e_N(\xi)h_N(\eta) \end{bmatrix} \mathbb{E}^{1,0} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} . \end{aligned} \tag{47}$$

Here $\mathbb{E}^{1,0}$ is called an *incidence matrix* which contains the values -1, 0 and 1. This incidence matrix is very sparse.

Let the space $\mathbf{D}^h$ be the span of $\{h_i(\xi)e_j(\eta)\} \times \{e_k(\xi)h_l(\eta)\}$, for $i,l = 0,\ldots,N$ and $j,k = 1,\ldots,N$, then (47) shows that $\nabla\times\ :\ \mathbf{C}^h \to \mathbf{D}^h$.

Let $\boldsymbol{f}^h \in \mathbf{D}^h$, then it can be expanded as

$$\boldsymbol{f}^h = \begin{pmatrix} \sum_{i=0}^{N} \sum_{j=1}^{N} f^{\xi}_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} f^{\eta}_{i,j} e_i(\xi) h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_0(\xi)e_1(\eta) & \dots & h_N(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_0(\eta) & \dots & e_N(\xi)h_N(\eta) \end{bmatrix} \begin{bmatrix} f^{\xi}_{0,1} \\ \vdots \\ f^{\xi}_{N,N} \\ f^{\eta}_{1,0} \\ \vdots \\ f^{\eta}_{N,N} \end{bmatrix}. \tag{48}$$

Since both $\nabla \times \boldsymbol{\omega}^h \in \mathbf{D}^h$ and $\boldsymbol{f}^h \in \mathbf{D}^h$, the equality $\nabla \times \boldsymbol{\omega}^h = \boldsymbol{f}^h$ makes sense in $\mathbf{D}^h$. If we equate (47) and (48), we obtain

$$\mathbb{E}^{1,0} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} = \begin{bmatrix} f^{\xi}_{0,1} \\ \vdots \\ f^{\xi}_{N,N} \\ f^{\eta}_{1,0} \\ \vdots \\ f^{\eta}_{N,N} \end{bmatrix}, \tag{49}$$

independently of the basis functions! So $\nabla\times$ can be discretized by the sparse incidence matrix $\mathbb{E}^{1,0}$ which only contains the entries -1, 0 and 1. This incidence matrix is *independent* of the mesh width, the shape of the mesh – you can stretch, twist, shear the grid but the incidence matrix remains the same – and it is independent of the order of the scheme. $\mathbb{E}^{1,0}$ represents the purely topological part of the derivative. All metric properties, size and shape of the grid and the order of the scheme, are contained in the basis functions.

The space $\mathbf{D}^h_0$ is obtained from $\mathbf{D}^h$ by setting all fluxes over the outer boundary to zero. An element $\boldsymbol{u}^h \in \mathbf{D}^h_0$ is therefore represented as

$$\boldsymbol{u}^h = \begin{pmatrix} \sum_{i=1}^{N-1} \sum_{j=1}^{N} u^{\xi}_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=1}^{N-1} u^{\eta}_{i,j} e_i(\xi) h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_1(\xi)e_1(\eta) & \dots & h_{N-1}(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_1(\eta) & \dots & e_N(\xi)h_{N-1}(\eta) \end{bmatrix} \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\xi}_{N-1,N} \\ u^{\eta}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix}. \tag{50}$$

Note that if $\boldsymbol{\psi}^h \in \mathbf{C}_0^h$ as in (46) then $\nabla \times \boldsymbol{\psi}^h \in \mathbf{D}_0^h$ therefore $\nabla \times \boldsymbol{\psi}^h = \boldsymbol{u}^h$ is well-defined in $\mathbf{D}_0^h$. The relation between the expansion coefficients of $\boldsymbol{\psi}^h$ and $\boldsymbol{u}^h$ is given by

$$\bar{\mathbb{E}}^{1,0} \begin{bmatrix} \psi_{1,1} \\ \vdots \\ \psi_{N-1,N-1} \end{bmatrix} = \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\xi}_{N-1,N} \\ u^{\eta}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix} . \tag{51}$$

Here $\bar{\mathbb{E}}^{1,0}$ is obtained from the $\mathbb{E}^{1,0}$ in (49) by eliminating the the rows which correspond to zero velocity fluxes over the outer boundary and the columns corresponding to the zero stream function along the outer boundary.

Let $\boldsymbol{f}^h \in \mathbf{D}^h$ as in (48) then the divergence of $\boldsymbol{f}^h$ is given by

$$\begin{aligned} \nabla \cdot \boldsymbol{f}^h &= \sum_{i=1}^{N} \sum_{j=1}^{N} \left(f^{\xi}_{i,j} - f^{\xi}_{i-1,j} + f^{\eta}_{i,j} - f^{\eta}_{i,j-1} \right) e_i(\xi) e_j(\eta) \\ &= [e_1(\xi)e_1(\eta) \quad \dots \quad e_N(\xi)e_N(\eta)] \mathbb{E}^{2,1} \begin{bmatrix} f^{\xi}_{0,1} \\ \vdots \\ f^{\xi}_{N,N} \\ f^{\eta}_{1,0} \\ \vdots \\ f^{\eta}_{N,N} \end{bmatrix} , \end{aligned} \tag{52}$$

where we used (42) again. The incidence matrix $\mathbb{E}^{2,1}$ is again a sparse matrix which only contains the values -1, 0 and 1. Note that for all $\boldsymbol{\omega}^h \in \mathbf{C}_0^h$ we have that

$$\nabla \cdot (\nabla \times \boldsymbol{\omega}^h) = [e_1(\xi)e_1(\eta) \quad \dots \quad e_N(\xi)e_N(\eta)] \mathbb{E}^{2,1} \mathbb{E}^{1,0} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} = 0 . \tag{53}$$

Since this has to hold for all $\boldsymbol{\omega}^h \in \mathbf{C}^h$ and the basis $\{e_i(\xi)e_j(\eta)\}$ is linearly independent, we need to have $\mathbb{E}^{2,1}\mathbb{E}^{1,0} \equiv 0$.

Let S^h be space spanned by the basis functions $\{e_i(\xi)e_j(\eta)\}$, then we see that $\nabla \cdot : \mathbf{D}^h \mapsto S^h$. So for any $q^h \in S^h$ expanded as

$$q^h = \sum_{i=1}^{N} \sum_{j=1}^{N} q_{i,j} e_i(\xi) e_j(\eta) , \tag{54}$$

the equation $\nabla \cdot \boldsymbol{f}^h = q^h$ makes sense in S^h and since both $\nabla \cdot \boldsymbol{f}^h$ and q^h are expanded in the same basis, the expansion coefficients need to be equal

$$\mathbb{E}^{2,1} \begin{bmatrix} f^{\xi}_{0,1} \\ \vdots \\ f^{\xi}_{N,N} \\ f^{\eta}_{1,0} \\ \vdots \\ f^{\eta}_{N,N} \end{bmatrix} = \begin{bmatrix} q_{1,1} \\ \vdots \\ q_{N,N} \end{bmatrix} . \tag{55}$$

For $\boldsymbol{u}^h \in \mathbf{D}^h_0$ the expansion of $\nabla \cdot \boldsymbol{u}^h$ has a similar expansion as (52)

$$\begin{aligned} \nabla \cdot \boldsymbol{u}^h &= \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u^{\xi}_{i,j} - u^{\xi}_{i-1,j} + u^{\eta}_{i,j} - u^{\eta}_{i,j-1} \right) e_i(\xi) e_j(\eta) \\ &= [e_1(\xi)e_1(\eta) \quad \dots \quad e_N(\xi)e_N(\eta)] \bar{\mathbb{E}}^{2,1} \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\xi}_{N-1,N} \\ u^{\eta}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix} , \end{aligned} \tag{56}$$

where $u^{\xi}_{0,j} = u^{\xi}_{N,j} = u^{\eta}_{i,0} = u^{\eta}_{i,N} = 0$ in the first equality. Note that the incidence matrix $\bar{\mathbb{E}}^{2,1}$ is obtained from $\mathbb{E}^{2,1}$ in (52) by removing the columns which correspond to the zero fluxes along the outer boundary.

Because $\nabla\cdot \, : \, \mathbf{D}^h_0 \to S^h_0$, we set $S^h_0 = \mathrm{span}\{e_i(\xi)e_j(\eta)\}$ for $i, j = 1, \dots, N$. This means that every $m^h \in S^h_0$ can be expanded as

$$m^h(\xi, \eta) = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{i,j} e_i(\xi) e_j(\eta) . \tag{57}$$

Using (40) we see that

$$m_{i,j} = \int_{\xi_{i-1}}^{\xi_i} \int_{\eta_{j-1}}^{\eta_j} m^h(\xi, \eta) \, \mathrm{d}\xi \mathrm{d}\eta .$$

The expansion coefficients $m_{i,j}$ therefore denote the integral of m^h over the two-dimensional volume $[\xi_{i-1}, \xi_i] \times [\eta_{j-1}, \eta_j]$.

If $m^h \in S^h_0$ is in the range of $\nabla\cdot$ applied to $\mathbf{D}^h_0$, i.e. $m^h = \nabla \cdot \boldsymbol{u}^h$ for $\boldsymbol{u}^h \in \mathbf{D}^h_0$ then

$$\int_{-1}^{1} \int_{-1}^{1} m^h \, \mathrm{d}\xi \mathrm{d}\eta = \int_{-1}^{1} \int_{-1}^{1} \nabla \cdot \boldsymbol{u}^h \, \mathrm{d}\xi \mathrm{d}\eta = 0 . \tag{58}$$

So, S^h_0 is obtained from S^h by imposing the constraint (58). Using (57) this implies that the degrees of freedom (expansion coefficients) $m_{i,j}$ need to satisfy

$$\sum_{i=1}^{N} \sum_{j=1}^{N} m_{i,j} = 0 . \tag{59}$$

For all $\boldsymbol{\psi}^h \in \mathbf{C}^h_0$ we have that $\nabla \cdot (\nabla \times \boldsymbol{\psi}^h) \equiv 0$ which implies analogous to (53) that $\bar{\mathbb{E}}^{2,1} \bar{\mathbb{E}}^{1,0} \equiv 0$.

Note also that conservation of mass, $\nabla \cdot \boldsymbol{u}^h = 0$, by (56) reduces to the following relation for the expansion coefficients of $\boldsymbol{u}^h$

$$\bar{\mathbb{E}}^{2,1} \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\xi}_{N-1,N} \\ u^{\eta}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix} = 0 \; . \tag{60}$$

We now have the conforming finite dimensional function spaces $\mathbf{C}^h \subset H(curl,\Omega)$, $\mathbf{D}^h \subset H(div,\Omega)$, $S^h \in L^2(\Omega)$, $\mathbf{C}^h_0 \subset H_0(curl,\Omega)$, $\mathbf{D}^h_0 \subset H_0(div,\Omega)$ and $S^h_0 \in L^2_0(\Omega)$, such that $\nabla\times$ and $\nabla\cdot$ form an exact sequence between these spaces, i.e we have the finite dimensional analogue of (6)

$$\mathcal{R}^h_0(\nabla\times) = \mathcal{N}^h_0(\nabla\cdot) \quad \text{and} \quad \mathcal{R}^h(\nabla\times) = \mathcal{N}^h(\nabla\cdot) \; . \tag{61}$$

This relation can also be expressed in terms of the incidence matrices as

$$\mathcal{R}^h(\bar{\mathbb{E}}^{1,0}) = \mathcal{N}^h(\bar{\mathbb{E}}^{2,1}) \quad \text{and} \quad \mathcal{R}^h(\mathbb{E}^{1,0}) = \mathcal{N}^h(\mathbb{E}^{2,1}) \; . \tag{62}$$

5.3. **Inner products.** In order to define the adjoint operators, we need to introduce inner-products on the various spaces.

5.3.1. *Inner-product on* $\mathbf{C}^h$. Let $\boldsymbol{\varphi}^h, \boldsymbol{\omega} \in \mathbf{C}^h$, then the L^2-inner product in $\mathbf{C}^h$ is given by

$$\left(\boldsymbol{\varphi}^h, \boldsymbol{\omega}^h\right)_0 = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{\varphi}^h \boldsymbol{\omega}^h \,\mathrm{d}\xi\mathrm{d}\eta = [\varphi_{0,0} \;\ldots\; \varphi_{N,N}]\, \mathbb{M}^{(0)} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} , \tag{63}$$

where $\mathbb{M}^{(0)}$ is an $(N+1)^2 \times (N+1)^2$ matrix with entries

$$\mathbb{M}^{(0)} = \int_{-1}^{1}\int_{-1}^{1} h_i(\xi)h_j(\eta)h_k(\xi)h_l(\eta)\,\mathrm{d}\xi\mathrm{d}\eta \;, \quad i,j,k,l = 0\ldots,N \; . \tag{64}$$

If we evaluate the integrals in the mass matrix using GLL integration, (43), and use (36), we see that $\mathbb{M}^{(0)}$ is a diagonal matrix with the product of the integration weights (44) on the diagonal.

5.3.2. *Inner-product on* $\mathbf{C}^h_0$. Let $\boldsymbol{\xi}^h, \psi \in \mathbf{C}^h_0$, then the L^2-inner product in $\mathbf{C}^h_0$ is given by

$$\left(\boldsymbol{\xi}^h, \boldsymbol{\psi}^h\right)_0 = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{\xi}^h \boldsymbol{\psi}^h \,\mathrm{d}\xi\mathrm{d}\eta = [\xi_{1,1} \;\ldots\; \varphi_{N-1,N-1}]\, \bar{\mathbb{M}}^{(0)} \begin{bmatrix} \psi_{1,1} \\ \vdots \\ \omega_{N-1,N-1} \end{bmatrix} , \tag{65}$$

where $\bar{\mathbb{M}}^{(0)}$ is an $(N-1)^2 \times (N-1)^2$ obtained from $\mathbb{M}^{(0)}$ by deleting the rows and columns corresponding to the prescribed zero values of $\boldsymbol{\xi}^h$ and $\boldsymbol{\psi}^h$ along the boundary. The mass matrix $\bar{\mathbb{M}}^{(0)}$ is still diagonal.

5.3.3. *Inner-product on* $\mathbf{D}^h$. For $\boldsymbol{u}^h, \boldsymbol{v}^h \in \mathbf{D}^h$ the inner-product is given by

$$\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right)_0 = \int_{-1}^{1}\int_{-1}^{1}(\boldsymbol{u}^h\boldsymbol{v}^h), \mathrm{d}\xi\mathrm{d}\eta = \left[u^{\xi}_{0,1} \dots u^{\eta}_{N,N}\right]\mathbb{M}^{(1)}\begin{bmatrix} v^{\xi}_{0,1} \\ \vdots \\ v^{\eta}_{N,N}\end{bmatrix} . \tag{66}$$

Here the mass matrix is given by

$$\mathbb{M}^{(1)} = \begin{pmatrix} \left(h_i(\xi)e_j(\eta)h_k(\xi)e_l(\eta)\right)_0 & 0 \\ 0 & \left(e_p(\xi)h_q(\eta)e_r(\xi)h_s(\eta)\right)_0 \end{pmatrix} , \tag{67}$$

for $i,k,q,s = 0,\dots,N$ and $j,l,q,r = 1,\dots,N$. The mass matrix $\mathbb{M}^{(1)}$ is a $2N(N+1) \times 2N(N+1)$ block diagonal matrix, but it is not diagonal.

5.3.4. *Inner-product on* $\mathbf{D}^h_0$. The mass matrix $\bar{\mathbb{M}}^{(1)}$ on $\mathbf{D}^h_0$ is obtained from $\mathbb{M}^{(1)}$ by removing the row and columns which correspond to the zero fluxes on the boundary. $\bar{\mathbb{M}}^{(1)}$ is then a $2N(N-1) \times 2N(N-1)$ block diagonal matrix.

5.3.5. *Inner-product on* S^h *and* S^h_0. The mass matrix $\mathbb{M}^{(2)}$ for both S^h and S^h_0 is given by

$$\mathbb{M}^{(2)} = \int_{-1}^{1}\int_{-1}^{1} e_i(\xi)e_j(\eta)e_k(\xi)e_l(\eta)\,\mathrm{d}\xi\mathrm{d}\eta , \tag{68}$$

for $i,j,k,l = 1,\dots,N$. $\mathbb{M}^{(2)}$ is not diagonal, but diagonal dominant.

5.4. **Finite dimensional adjoint operators.** With the primary operators $\nabla\times$ and $\nabla\cdot$ defined in Section 5.2.1 and the inner-products in Section 5.3, we can now define the adjoint operators in the same way as in Section 2.2.

Let $\boldsymbol{\psi}^h \in \mathbf{C}^h_0$, then $\nabla\times\boldsymbol{\psi}^h \in \mathbf{D}^h_0$, if we take the inner-product with any $\boldsymbol{v}^h \in \mathbf{D}^h_0$ we have in terms of the expansion coefficients of $\boldsymbol{\psi}^h$ and $\boldsymbol{v}^h$

$$\begin{aligned}
\left(\boldsymbol{v}^h, \nabla\times\boldsymbol{\psi}^h\right)_0 &= \left[v^{\xi}_{1,1} \dots v^{\eta}_{N,N-1}\right]\bar{\mathbb{M}}^{(1)}\bar{\mathbb{E}}^{(1,0)}\begin{bmatrix}\psi_{1,1} \\ \vdots \\ \psi_{N-1,N-1}\end{bmatrix} \\
&= \left[v^{\xi}_{1,1} \dots v^{\eta}_{N,N-1}\right]\bar{\mathbb{M}}^{(1)}\bar{\mathbb{E}}^{(1,0)}\bar{\mathbb{M}}^{(0)^{-1}}\bar{\mathbb{M}}^{(0)}\begin{bmatrix}\psi_{1,1} \\ \vdots \\ \psi_{N-1,N-1}\end{bmatrix} \\
&= \left(\nabla^*\times\boldsymbol{v}^h, \boldsymbol{\psi}^h\right) .
\end{aligned} \tag{69}$$

So the expansion coefficients of $\nabla^*\times\boldsymbol{v}^h$ are given by

$$\bar{\mathbb{M}}^{(0)^{-1}}\bar{\mathbb{E}}^{(1,0)^T}\bar{\mathbb{M}}^{(1)}\begin{bmatrix} v^{\xi}_{1,1} \\ \vdots \\ v^{\eta}_{N,N-1}\end{bmatrix} .$$

These are the expansion coefficients in $\mathbf{C}^h_0$. So if $\nabla\times : \mathbf{C}^h_0 \to \mathbf{D}^h_0$ then $\nabla^*\times : \mathbf{D}^h_0 \to \mathbf{C}^h_0$. Note that in this integration by parts no boundary integral is neglected because $\boldsymbol{\psi}^h$ vanishes along the boundary.

Let $\boldsymbol{u}^h \in \mathbf{D}_0^h$ then $\nabla \cdot \boldsymbol{u}^h \in S_0^h$. If we take the inner-product with any $\phi \in S_0^h$, we obtain in terms of the expansion coefficients of $\boldsymbol{u}^h$ and ϕ^h

$$\begin{aligned}\left(\phi^h, \nabla \cdot \boldsymbol{u}^h\right)_0 &= \left[\phi_{1,1} \ \ \ldots \ \ \phi_{N,N}\right] \bar{\mathbb{M}}^{(2)} \bar{\mathbb{E}}^{2,1} \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix} \\ &= \left[\phi_{1,1} \ \ \ldots \ \ \phi_{N,N}\right] \bar{\mathbb{M}}^{(2)} \bar{\mathbb{E}}^{2,1} \bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{M}}^{(1)} \begin{bmatrix} u^{\xi}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{bmatrix} \\ &= \left(\nabla^* \phi, \boldsymbol{u}^h\right)_0 \ . \end{aligned} \tag{70}$$

Therefore, the expansion coefficients of $\nabla^* \phi \in \mathbf{D}_0^h$ are given by

$$\bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{2,1^T} \bar{\mathbb{M}}^{(2)} \begin{bmatrix} \phi_{1,1} \\ \vdots \\ \phi_{N,N} \end{bmatrix} .$$

These are expansion coefficients in $\mathbf{D}_0^h$ so we use the basis functions in $\mathbf{D}_0^h$ to expand $\nabla^* \phi^h$, such that $\nabla^* : S_0^h \rightarrow \mathbf{D}_0^h$. Note that again no boundary integrals were neglected, because $\boldsymbol{u}^h \cdot \boldsymbol{n} = 0$ for all $\boldsymbol{u}^h \in \mathbf{D}_0^h$.

From

$$0 = \left(\nabla \cdot (\nabla \times \psi^h), \phi^h\right)_0 = \left(\nabla \times \psi^h, \nabla^* \phi^h\right)_0 = \left(\psi^h, \nabla^* \times (\nabla^* \phi^h)\right)_0 \ ,$$

it follows that

$$\mathcal{R}_0^h(\nabla^*) = \mathcal{N}_0^h(\nabla^* \times) \quad \text{ and } \quad \mathcal{R}_0^h(\nabla \times) \perp \mathcal{R}_0^h(\nabla^*) \ .$$

This can also be seen from the expansion coefficients. The expansion coefficients $\vec{\phi}$ of ϕ^h are mapped on the expansion coefficients $\bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{2,1^T} \bar{\mathbb{M}}^{(2)} \vec{\phi}$ of $\nabla^* \phi^h$, which are then mapped by to the expansion coefficients $\bar{\mathbb{M}}^{(0)^{-1}} \bar{\mathbb{E}}^{(1,0)^T} \bar{\mathbb{M}}^{(1)} \bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{2,1^T} \bar{\mathbb{M}}^{(2)} \vec{\phi}$ of $\nabla^* \times \nabla^* \phi^h$ which is zero because $\bar{\mathbb{E}}^{(1,0)^T} \bar{\mathbb{E}}^{2,1^T} = \left(\bar{\mathbb{E}}^{(1,0)} \bar{\mathbb{E}}^{2,1}\right) \equiv 0$.

We can now write the orthogonal decomposition of any $\boldsymbol{u}^h \in \mathbf{D}_0^h$ in terms of the expansion coefficients. Let $\psi^h \in \mathbf{C}_0^h$ and $\phi^h \in S_0^h$ such that

$$\boldsymbol{u}^h = \nabla \times \psi^h + \nabla^* \phi^h \ .$$

Then we have for the expansion coefficients

$$\vec{u} = \bar{\mathbb{E}}^{1,0} \vec{\psi} + \bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{(2,1)^T} \bar{\mathbb{M}}^{(2)} \vec{\phi} \ ,$$

from which we can obtain the expansion coefficients $\vec{\psi}$ and $\vec{\phi}$ by solving the symmetric systems

$$\bar{\mathbb{M}}^{(2)} \bar{\mathbb{E}}^{2,1} \vec{u} = \bar{\mathbb{M}}^{(2)} \bar{\mathbb{E}}^{2,1} \bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{(2,1)^T} \bar{\mathbb{M}}^{(2)} \vec{\phi} \quad \text{ and } \quad \bar{\mathbb{E}}^{1,0^T} \bar{\mathbb{M}}^{(1)} \vec{u} = \bar{\mathbb{E}}^{1,0^T} \bar{\mathbb{M}}^{(1)} \bar{\mathbb{E}}^{1,0} \vec{\psi} \ .$$

$$\mathbf{C}_0^h \underset{\nabla^* \times}{\overset{\nabla \times}{\rightleftarrows}} \mathbf{D}_0^h \underset{\nabla^*}{\overset{\nabla \cdot}{\rightleftarrows}} S_0^h$$

Or in terms of the expansion coefficients in these space

$$\mathfrak{E}(\mathbf{C}_0^h) \underset{\bar{\mathbb{M}}^{(0)^{-1}} \bar{\mathbb{E}}^{1,0^T} \bar{\mathbb{M}}^{(1)}}{\overset{\bar{\mathbb{E}}^{1,0}}{\rightleftarrows}} \mathfrak{E}(\mathbf{D}_0^h) \underset{\bar{\mathbb{M}}^{(1)^{-1}} \bar{\mathbb{E}}^{2,1^T} \bar{\mathbb{M}}^{(2)}}{\overset{\bar{\mathbb{E}}^{2,1}}{\rightleftarrows}} \mathfrak{E}(S_0^h)$$

A similar construction can be applied for the adjoint operators between the spaces $\mathbf{C}^h$, $\mathbf{D}^h$ and S^h. Let $\boldsymbol{\omega}^h \in \mathbf{C}^h$, then $\nabla \times \boldsymbol{\omega}^h \in \mathbf{D}^h$. If we take an arbitrary $\boldsymbol{v}^h \in \mathbf{D}^h$, then the inner product in terms of the expansion coefficients can be written as

$$\begin{aligned}
\left(\boldsymbol{v}^h, \nabla \times \boldsymbol{\omega}^h\right)_0 &= \left[v^\xi_{0,1} \dots v^\eta_{N,N}\right] \mathbb{M}^{(1)}\mathbb{E}^{(1,0)} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} \\
&= \left[v^\xi_{0,1} \dots v^\eta_{N,N}\right] \mathbb{M}^{(1)}\mathbb{E}^{(1,0)}\mathbb{M}^{(0)^{-1}}\mathbb{M}^{(0)} \begin{bmatrix} \omega_{0,0} \\ \vdots \\ \omega_{N,N} \end{bmatrix} \\
&= \left(\nabla^* \times \boldsymbol{v}^h, \boldsymbol{\omega}^h\right) . \qquad (71)
\end{aligned}$$

So the expansion coefficients of $\nabla^* \times \boldsymbol{v}^h$ are given by

$$\mathbb{M}^{(0)^{-1}}\mathbb{E}^{(1,0)^T}\mathbb{M}^{(1)} \begin{bmatrix} v^\xi_{0,1} \\ \vdots \\ v^\eta_{N,N} \end{bmatrix} .$$

Note that in (71) we used (13), which implies that we implicitly set $\boldsymbol{v}^h \times \boldsymbol{n} = 0$. The expansion coefficients of $\nabla^* \times \boldsymbol{v}^h$ are expanded with the basis functions in $\mathbf{C}^h$, therefore $\nabla^*\times : \mathbf{D}^h \to \mathbf{C}^h$

For $\boldsymbol{u}^h \in \mathbf{D}^h$ then $\nabla \cdot \boldsymbol{u}^h \in S^h$. If we take the inner-product with any $p^h \in S^h$, we obtain in terms of the expansion coefficients of $\boldsymbol{u}^h$ and p^h

$$\begin{aligned}
\left(p^h, \nabla \cdot \boldsymbol{u}^h\right)_0 &= \left[p_{1,1} \quad \dots \quad p_{N,N}\right] \mathbb{M}^{(2)}\mathbb{E}^{2,1} \begin{bmatrix} u^\xi_{0,1} \\ \vdots \\ u^\eta_{N,N} \end{bmatrix} \\
&= \left[p_{1,1} \quad \dots \quad p_{N,N}\right] \mathbb{M}^{(2)}\mathbb{E}^{2,1}\mathbb{M}^{(1)^{-1}}\mathbb{M}^{(1)} \begin{bmatrix} u^\xi_{0,1} \\ \vdots \\ u^\eta_{N,N} \end{bmatrix} \\
&= \left(\nabla^* p^h, \boldsymbol{u}^h\right)_0 . \qquad (72)
\end{aligned}$$

Therefore, the expansion coefficients of $\nabla^* p^h \in \mathbf{D}^h$ are given by

$$\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T}\mathbb{M}^{(2)} \begin{bmatrix} p_{1,1} \\ \vdots \\ p_{N,N} \end{bmatrix} .$$

These are expansion coefficients in $\mathbf{D}^h$ so we use the basis functions in $\mathbf{D}^h$ to expand $\nabla^* p^h$, such that $\nabla^* : S^h \to \mathbf{D}^h$. In (72), we neglected boundary integrals using (14) which weakly impose $\boldsymbol{u}^h \cdot \boldsymbol{n} = 0$ on functions in $\mathbf{D}^h$.

With these formal adjoints (neglecting boundary integrals) we have that

$$\mathcal{R}^h(\nabla^*) = \mathcal{N}^h(\nabla^*\times) ,$$

however $\mathcal{R}^h(\nabla\times) \perp \mathcal{R}^h(\nabla^*)$ is no longer valid, see the introduction of Section 3.

$$\mathbf{C}^h \underset{\nabla^*\times}{\overset{\nabla\times}{\rightleftarrows}} \mathbf{D}^h \underset{\nabla^*}{\overset{\nabla\cdot}{\rightleftarrows}} S^h$$

Or in terms of the expansion coefficients in these space

$$\mathfrak{E}(\mathbf{C}^h) \underset{\mathbb{M}^{(0)^{-1}}\mathbb{E}^{1,0^T}\mathbb{M}^{(1)}}{\overset{\mathbb{E}^{1,0}}{\rightleftarrows}} \mathfrak{E}(\mathbf{D}^h) \underset{\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T}\mathbb{M}^{(2)}}{\overset{\mathbb{E}^{2,1}}{\rightleftarrows}} \mathfrak{E}(S^h)$$

5.5. **Mimetic least-squares for the Stokes problem.** With the conforming finite dimensional spaces and the primary and adjoint vector operations between these spaces, we are now in the position to implement (35).

$$\begin{aligned}
\left(\nabla\times\boldsymbol{\omega}^h,\nabla\times\boldsymbol{\xi}^h\right)_0+\left(\boldsymbol{\omega}^h,\boldsymbol{\xi}^h\right)_0-\left(\boldsymbol{u}^h,\nabla\times\boldsymbol{\xi}^h\right)_0 &= \left(\boldsymbol{f},\nabla\times\boldsymbol{\xi}^h\right)_0 && \forall\boldsymbol{\xi}^h\in\mathbf{C}^h\\
-\left(\nabla\times\boldsymbol{\omega}^h,\boldsymbol{v}^h\right)_0+\left(\nabla^*\times\boldsymbol{u}^h,\nabla^*\times\boldsymbol{v}^h\right)_0+\left(\nabla\cdot\boldsymbol{u}^h,\nabla\cdot\boldsymbol{v}^h\right)_0 &= 0 && \forall\boldsymbol{v}^h\in\mathbf{D}^h_0\\
\left(\nabla^*p^h,\nabla^*q^h\right)_0 &= \left(-\nabla\cdot\boldsymbol{f},q^h\right)_0 && \forall q^h\in S^h
\end{aligned}$$

We will use the representation of vector operations and their associated adjoint operations on the expansion coefficients and the inner-products on the various spaces. Since some operations are multiply defined, depending to which sequence they belong it is important to establish in which sequence the operations take place; the one with boundary conditions $\mathbf{C}^h_0\to\mathbf{D}^h_0\to S^h_0$ or the sequence without boundary conditions $\mathbf{C}^h\to\mathbf{D}^h\to S^h$.

The term $\left(\nabla\times\boldsymbol{\omega}^h,\nabla\times\boldsymbol{\xi}^h\right)_0$ is an inner-product in $\mathbf{D}^h$ and is represented as

$$[\xi_{0,0}\quad\dots\quad\xi_{N,N}]\mathbb{E}^{1,0^T}\mathbb{M}^{(1)}\mathbb{E}^{1,0}\begin{bmatrix}\omega_{0,0}\\ \vdots\\ \omega_{N,N}\end{bmatrix}.$$

The term $\left(\boldsymbol{\omega}^h,\boldsymbol{\xi}^h\right)_0$ is an inner-product on $\mathbf{C}^h$ and will be represented as

$$[\xi_{0,0}\quad\dots\quad\xi_{N,N}]\mathbb{M}^{(0)}\begin{bmatrix}\omega_{0,0}\\ \vdots\\ \omega_{N,N}\end{bmatrix}.$$

The term $\left(\nabla^*\times\boldsymbol{u}^h,\nabla^*\times\boldsymbol{v}^h\right)_0$ is an inner-product on $\mathbf{C}^h_0$ and will therefore be represented as

$$[v^\xi_{1,1}\quad\dots\quad v^\eta_{N,N-1}]\bar{\mathbb{M}}^{(1)}\bar{\mathbb{E}}^{1,0}\bar{\mathbb{M}}^{(0)^{-1}}\bar{\mathbb{E}}^{1,0^T}\bar{\mathbb{M}}^{(1)}\begin{bmatrix}u^\xi_{1,1}\\ \vdots\\ u^\eta_{N,N-1}\end{bmatrix}.$$

The term $\left(\nabla^*p^h,\nabla^*q^h\right)_0$ is an inner-product on $\mathbf{D}^h$ and will therefore be discretized as

$$[q_{1,1}\quad\dots\quad q_{N,N}]\mathbb{M}^{(2)}\mathbb{E}^{2,1}\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T}\mathbb{M}^{(2)}\begin{bmatrix}p_{1,1}\\ \vdots\\ p_{N,N}\end{bmatrix}.$$

Mass conservation, as represented by the term $\left(\nabla\cdot\boldsymbol{u}^h,\nabla\cdot\boldsymbol{v}^h\right)_0$ is an inner-product on S^h_0 and will therefore be represented by

$$[v^\xi_{1,1}\quad\dots\quad v^\eta_{N,N-1}]\bar{\mathbb{E}}^{2,1^T}\mathbb{M}^{(2)}\bar{\mathbb{E}}^{2,1}\begin{bmatrix}u^\xi_{1,1}\\ \vdots\\ u^\eta_{N,N-1}\end{bmatrix}.$$

The term $\left(\boldsymbol{u}^h, \nabla \times \boldsymbol{\xi}^h\right)_0$ deviates from the other terms in the sense that it is in an inner-product on $\mathbf{D}^h$, since $\nabla \times \boldsymbol{\xi}^h$ for $\boldsymbol{\xi}^h \in \mathbf{C}^h$ maps into $\mathbf{D}^h$. The space $\mathbf{D}^h$ is spanned by $2N(N+1)$ basis functions if polynomials of degree N are used. We then take the bilinear product with $\boldsymbol{u}^h \in \mathbf{D}_0^h$ which is spanned by $2N(N-1)$ basis functions. This means that the "mass matrix" is non-square. This term will be represented as

$$[\xi_{0,0} \quad \dots \quad \xi_{N,N}]\mathbb{E}^{1,0^T}\mathbb{M}^{(1)^\star}\left[\begin{array}{c} u^{\xi}_{1,1} \\ \vdots \\ u^{\eta}_{N,N-1} \end{array}\right],$$

where $\mathbb{M}^{(1)^\star}$ is the $2N(N+1) \times 2N(N-1)$ matrix obtained from $\mathbb{M}^{(1)}$ by eliminating the columns corresponding to prescribed zero fluxes in $\mathbf{D}_0^h$.

The right hand side term $\left(\boldsymbol{f}, \nabla \times \boldsymbol{\xi}^h\right)_0$ is an inner-product on $\mathbf{C}^h$ which is represented as

$$[\xi_{0,0} \quad \dots \quad \xi_{N,N}]\mathbb{E}^{1,0^T}\mathbb{M}^{(1)}\left[\begin{array}{c} f^{\xi}_{0,1} \\ \vdots \\ f^{\eta}_{N,N} \end{array}\right],$$

and the terms $\left(-\nabla \cdot \boldsymbol{f}, q^h\right)_0$ is an inner product in S^h represented as

$$[q_{1,1} \quad \dots \quad q_{N,N}]\mathbb{M}^{(2)}\mathbb{E}^{2,1}\left[\begin{array}{c} f^{\xi}_{0,1} \\ \vdots \\ f^{\eta}_{N,N} \end{array}\right].$$

If we collect all these contributions, the mimetic least-squares formulation with no-slip boundary conditions becomes

(73)
$$\begin{pmatrix} \mathbb{M}^{(0)} + \mathbb{E}^{1,0^T}\mathbb{M}^{(1)}\mathbb{E}^{1,0} & -\mathbb{E}^{1,0^T}\mathbb{M}^{(1)^\star} & 0 \\ -\mathbb{M}^{(1)^{\star T}}\mathbb{E}^{1,0} & \bar{\mathbb{M}}^{(1)}\bar{\mathbb{E}}^{1,0}\bar{\mathbb{M}}^{(0)^{-1}}\bar{\mathbb{E}}^{1,0^T}\bar{\mathbb{M}}^{(1)} + \bar{\mathbb{E}}^{2,1^T}\mathbb{M}^{(2)}\bar{\mathbb{E}}^{2,1} & 0 \\ 0 & 0 & \mathbb{M}^{(2)}\mathbb{E}^{2,1}\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T}\mathbb{M}^{(2)} \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbb{E}^{1,0^T}\mathbb{M}^{(0)}\boldsymbol{f}^h \\ 0 \\ -\mathbb{M}^{(2)}\mathbb{E}^{2,1}\boldsymbol{f}^h \end{pmatrix}.$$

The final discrete system consists only of mass matrices and incidence matrices related to the various space and operations between these spaces. We started this section by stating that a mimetic method aims to decompose a PDE in a purely topological part and a metric dependent part. The mimetic least-squares formulation precisely achieves this, where the topological part is represented by the incidence matrices and the metric dependent part by the mass matrices. If the mesh is deformed, the mass matrices will change, but the incidence matrices will remain the same.

6. Numerical results

In order to asses the performance of the newly developed mimetic least-squares formulation, we present several test problems.

6.1. **Test case 1.** Consider the Stokes problem defined on the domain $\Omega = [-1,1]^2$ with $\boldsymbol{u} = 0$ along the boundary. We take as exact velocity field

(74)
$$\boldsymbol{u}(x,y) = \begin{pmatrix} -4y(1-y^2)(1-x^2)^2\sin(2\pi(x+y)) + 2\pi(1-x^2)^2(1-y^2)^2\cos(2\pi(x+y)) \\ 4x(1-x^2)(1-y^2)^2\sin(2\pi(x+y)) - 2\pi(1-x^2)^2(1-y^2)^2\cos(2\pi(x+y)) \end{pmatrix}.$$

This velocity field is divergence free. For the right hand side function $\boldsymbol{f}$ we take $\boldsymbol{f} = \nabla^* \times \nabla \times \boldsymbol{u}$. In this case the corresponding exact pressure field is constant.

N	4	6	8	10	12	14	16	18
$\mathcal{J}(\omega^h, \boldsymbol{u}^h, p^h; \boldsymbol{f})$	$2.6 \cdot 10^4$	$5.7 \cdot 10^3$	$4.4 \cdot 10^3$	$1.6 \cdot 10^3$	$5.0 \cdot 10^1$	$5.5 \cdot 10^{-1}$	$2.8 \cdot 10^{-3}$	$7.7 \cdot 10^{-6}$
$\frac{1}{2}\|\omega^h - \nabla^* \times \boldsymbol{u}^h\|_0^2$	$1.3 \cdot 10^4$	$3.9 \cdot 10^2$	$5.7 \cdot 10^3$	$9.3 \cdot 10^2$	$3.2 \cdot 10^1$	$4.0 \cdot 10^{-1}$	$2.2 \cdot 10^{-3}$	$6.0 \cdot 10^{-6}$
$\frac{1}{2}\|\nabla \times \omega^h + \nabla^* p - f\|_D^2$	$1.3 \cdot 10^4$	$5.3 \cdot 10^3$	$1.3 \cdot 10^3$	$7.0 \cdot 10^2$	$1.8 \cdot 10^1$	$1.5 \cdot 10^{-1}$	$5.7 \cdot 10^{-4}$	$1.6 \cdot 10^{-6}$
$\frac{1}{2}\|\nabla \cdot \boldsymbol{u}^h\|_0^2$	$7.4 \cdot 10^{-14}$	$4.1 \cdot 10^{-14}$	$6.5 \cdot 10^{-13}$	$8.1 \cdot 10^{-14}$	$4.3 \cdot 10^{-14}$	$3.0 \cdot 10^{-13}$	$1.1 \cdot 10^{-13}$	$4.7 \cdot 10^{-13}$
$\|\omega^h - \omega\|_{H(curl,\Omega)}^2$	$3.2 \cdot 10^5$	$3.1 \cdot 10^5$	$9.6 \cdot 10^4$	$8.0 \cdot 10^3$	$2.1 \cdot 10^2$	$2.2 \cdot 10^0$	$1.1 \cdot 10^{-2}$	$3.2 \cdot 10^{-5}$
$\|\boldsymbol{u}^h - \boldsymbol{u}\|_{H_0(div,\Omega)}^2$	$2.9 \cdot 10^4$	$3.8 \cdot 10^3$	$1.2 \cdot 10^4$	$2.0 \cdot 10^3$	$6.6 \cdot 10^1$	$8.4 \cdot 10^{-1}$	$4.7 \cdot 10^{-3}$	$1.3 \cdot 10^{-5}$
$\|p^h - p\|_{L^2(\Omega)/\mathbb{R}}^2$	$1.0 \cdot 10^4$	$8.1 \cdot 10^1$	$9.7 \cdot 10^0$	$7.8 \cdot 10^{-1}$	$1.1 \cdot 10^{-2}$	$6.5 \cdot 10^{-5}$	$1.8 \cdot 10^{-7}$	$2.6 \cdot 10^{-10}$

TABLE 1. Convergence results for Test case 1 with increasing polynomial degree

In Table 1 we list the results for this test case as a function of N. In the second row the minimum of the least-squares functional is listed. The 3 residuals which make up the least-squares functional are presented in the following rows. The error in vorticity $\|\omega^h - \omega\|_{H(curl,\Omega)}^2$, the error in velocity is measured in the $\|\cdot\|_{H_0(div,\Omega)}$-norm and the error in the pressure is measured in the $\|\cdot\|_{L^2(\Omega)/\mathbb{R}}$-norm.

The first thing to note is that conservation of mass is satisfied up to machine precision. Secondly, we see that the pressure is not independent of vorticity and velocity – as would be the case for with normal velocity conditions and vorticity prescribed along the boundary.

6.2. **Test case 2.** Consider the Stokes problem defined on the domain $\Omega = [-1, 1]^2$ with $\boldsymbol{u} = 0$ along the boundary. The right hand side function $\boldsymbol{f}$ in this case is given by

$$f(x, y) = \begin{pmatrix} \pi \cos(\pi(x + y)) \\ \pi \cos(\pi(x + y)) \end{pmatrix} . \tag{75}$$

For this particular flow we expect a pressure field given by

$$p(x, y) = \sin(\pi(x + y)) + C , \tag{76}$$

where C is an arbitrary constant. The velocity and vorticity for this problem are identically zero.

N	4	6	8	10	12	14
$\mathcal{J}(\omega^h, \boldsymbol{u}^h, p^h; \boldsymbol{f})$	$8.4 \cdot 10^2$	$2.2 \cdot 10^0$	$6.9 \cdot 10^{-1}$	$1.5 \cdot 10^{-2}$	$2.5 \cdot 10^{-4}$	$2.8 \cdot 10^{-6}$
$\frac{1}{2}\|\omega^h - \nabla^* \times \boldsymbol{u}^h\|_0^2$	$6.8 \cdot 10^{-25}$	$2.2 \cdot 10^{-23}$	$8.6 \cdot 10^{-23}$	$2.1 \cdot 10^{-20}$	$2.3 \cdot 10^{-19}$	$4.6 \cdot 10^{-19}$
$\frac{1}{2}\|\nabla \times \omega^h + \nabla^* p - f\|_D^2$	$8.4 \cdot 10^2$	$2.2 \cdot 10^0$	$6.9 \cdot 10^{-1}$	$1.5 \cdot 10^{-2}$	$2.5 \cdot 10^{-4}$	$2.8 \cdot 10^{-6}$
$\frac{1}{2}\|\nabla \cdot \boldsymbol{u}^h\|_0^2$	$1.3 \cdot 10^{-26}$	$1.2 \cdot 10^{-24}$	$1.0 \cdot 10^{-23}$	$1.6 \cdot 10^{-21}$	$5.9 \cdot 10^{-19}$	$7.5 \cdot 10^{-20}$
$\|\omega^h - \omega\|_{H(curl,\Omega)}^2$	$4.0 \cdot 10^{-24}$	$1.5 \cdot 10^{-21}$	$2.9 \cdot 10^{-20}$	$1.7 \cdot 10^{-17}$	$5.2 \cdot 10^{-16}$	$2.4 \cdot 10^{-15}$
$\|\boldsymbol{u}^h - \boldsymbol{u}\|_{H_0(div,\Omega)}^2$	$1.4 \cdot 10^{-24}$	$5.2 \cdot 10^{-23}$	$2.4 \cdot 10^{-22}$	$5.3 \cdot 10^{-20}$	$6.0 \cdot 10^{-19}$	$1.3 \cdot 10^{-18}$
$\|p^h - p\|_{L^2(\Omega)/\mathbb{R}}^2$	$1.6 \cdot 10^3$	$3.6 \cdot 10^1$	$1.7 \cdot 10^{-1}$	$2.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-7}$	$3.6 \cdot 10^{-11}$

TABLE 2. Convergence results for Test case 2 with increasing polynomial degree

The results for test case 2 are listed in Table 2. We see that for all polynomial degrees we can capture exactly the zero velocity and vorticity field (up to machine accuracy) and therefore also mass conservation is satisfied up to machine precision. What is more striking in comparison to Test case 1 is that a non-zero velocity-vorticity field influences the pressure approximation, see Table 1. A non-constant pressure field does not influence, however, does not influence velocity-vorticity.

6.3. **Test case 3.** In Test case 1 we used a divergence-free right hand side function, while in Test case 2 the right hand side function was irrotational. In this test case, we combine these two cases by using the exact velocity field from Test case 1 and the exact pressure field from Test case 2. The corresponding right hand side function is then given by

$$\boldsymbol{f} = \nabla \times \nabla^* \times \boldsymbol{u} + \nabla^* p \; . \tag{77}$$

The results of this test case are displayed in Table 3

N	4	6	8	10	12	14
$\mathcal{J}(\omega^h, \boldsymbol{u}^h, p^h; \boldsymbol{f})$	$2.7 \cdot 10^4$	$5.7 \cdot 10^3$	$4.4 \cdot 10^3$	$1.6 \cdot 10^3$	$5.0 \cdot 10^1$	$5.5 \cdot 10^{-1}$
$\frac{1}{2}\|\omega^h - \nabla^* \times \boldsymbol{u}^h\|_0^2$	$1.3 \cdot 10^4$	$3.9 \cdot 10^2$	$5.7 \cdot 10^3$	$9.3 \cdot 10^2$	$3.2 \cdot 10^1$	$4.0 \cdot 10^{-1}$
$\frac{1}{2}\|\nabla \times \omega^h + \nabla^* p - f\|_D^2$	$1.4 \cdot 10^4$	$5.3 \cdot 10^3$	$1.2 \cdot 10^3$	$7.0 \cdot 10^2$	$1.8 \cdot 10^1$	$1.5 \cdot 10^{-1}$
$\frac{1}{2}\|\nabla \cdot \boldsymbol{u}^h\|_0^2$	$3.1 \cdot 10^{-13}$	$3.6 \cdot 10^{-14}$	$2.7 \cdot 10^{-13}$	$1.8 \cdot 10^{-13}$	$4.6 \cdot 10^{-13}$	$2.8 \cdot 10^{-13}$
$\|\omega^h - \omega\|_{H(curl,\Omega)}^2$	$3.3 \cdot 10^5$	$3.1 \cdot 10^5$	$9.6 \cdot 10^4$	$8.0 \cdot 10^3$	$2.1 \cdot 10^2$	$2.4 \cdot 10^{-15}$
$\|\boldsymbol{u}^h - \boldsymbol{u}\|_{H_0(div,\Omega)}^2$	$2.9 \cdot 10^4$	$3.8 \cdot 10^3$	$1.2 \cdot 10^4$	$2.0 \cdot 10^3$	$6.6 \cdot 10^1$	$8.4 \cdot 10^{-1}$
$\|p^h - p\|_{L^2(\Omega)/\mathbb{R}}^2$	$1.6 \cdot 10^4$	$1.2 \cdot 10^2$	$9.8 \cdot 10^0$	$7.8 \cdot 10^{-1}$	$1.1 \cdot 10^{-2}$	$6.6 \cdot 10^{-5}$

TABLE 3. Convergence results for Test case 3 with increasing polynomial degree

The convergence results for Test case 3 are very similar to those of Test case 1. While all residuals go to zero with increasing polynomial degree, conservation of mass, $\nabla \cdot \boldsymbol{u}^h = 0$, is satisfied up to machine accuracy for all polynomial degrees

In these 3 test cases the no-slip condition is enforced weakly without the need for adjustable parameters to enforce he no-slip constraint which is usually employed in least-squares finite element methods.

7. DISCUSSION

In this paper we developed a mimetic least-squares spectral element formulation for Stokes flow with no-slip (velocity) boundary conditions $\boldsymbol{u} = 0$ along the boundary of the domain.

Stokes flow with no-slip boundary conditions involves to exact sequences. One sequence consisting of the spaces $H_0(curl,\Omega)$, $H_0(curl,\Omega)$ and $L_0^2(\Omega)$ has boundary conditions, while the other sequence consisting of $H(curl,\Omega)$, $H(div,\Omega)$ and $L^2(\Omega)$ does not contain boundary conditions.

A non-standard stability proof for well-posedness is required in order to bound the momentum equation in $H(div,\Omega)$ from below to establish well-posedness of the least-squares formulation.

Conforming finite dimensional function spaces have been constructed in a spectral element context as well as the primary and adjoint operators between these spaces.

All these operations can be represented by operations on the expansion coefficients, i.e. on the degrees of freedom in the various functions spaces. These operations can be divided in topological operations by means of incidence matrices and metric-dependent operations represented by mass matrices.

Although the no-slip condition is weakly enforced in this formulation, there are no adjustable parameters to enforce the no-slip constraint.

Numerical tests for non-trivial right-hand side functions reveal that the method is convergent and that mass is conserved for all polynomial degrees. Approximation only takes place in the momentum equation and the definition of vorticity.

Future work will focus on multi-element methods, curvilinear grids and error estimates based on the current least-squares formalism.

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